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# FORCED VIBRATIONS OF ELASTIC BENDING–TORSION COUPLED BEAMS

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The objective of the present paper is to analyze coupled bending and torsional vibrations of distributed-parameter beams. The governing coupled set of partial differential equations is solved by separating the dynamic response in a quasistatic and in a complementary dynamic response. The quasistatic portion that may also contain singularities or discontinuities due to sudden load changes is determined in a closed form. The remaining complementary dynamic part is non-singular and can be approximated by a truncated modal series of fast accelerated convergence. The solution of the resulting generalized decoupled single-degree-of-freedom oscillators is given by means of Duhamel's convolution integral, whereby the acceleration of the loads is the driving term. The proposed procedure is illustrated for a dynamically loaded simply supported beam with channel cross-section, and the improvement in comparison to the classical modal analysis is demonstrated.


## 1. INTRODUCTION

The dynamic behavior of structural elements such as beams has received much attention and is the subject of many textbooks and papers. It is significant that most of the work reported in the literature deals with beams of doubly symmetric cross-section (centroid and shear axis are coincident), i.e., bending and torsion are decoupled. In many engineering structures such as beams with channel cross-sections, bridges, aircraft wings and propeller blades, however, the centroid and shear axis do not coincide which results in coupling between flexural vibrations and torsional vibrations.

For example, the free vibration analysis of simply supported beams with monosymmetric cross-sections can be found in the textbook of Weaver *et al*. [1]. Dokumaci [2] determined the eigenfrequencies of beams with other boundary conditions. Bishop *et al*. [3] extended these investigations to allow for warping of the cross-section and they showed that neglecting of warping makes a large difference to the eigenfrequencies of thin-walled beams with open cross-section. Coupled bending–torsional vibrations of Timoshenko beams were studied by Bishop and Price [4] and more recently, Bercin and Tanaka [5] took into account the warping stiffness to determine the free vibration modes of shear deformable beams. However, the number of investigations which study forced vibrations of

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coupled bending–torsional beams is rather limited. Recently, Eslimy-Isfahany *et al*. [6] calculated the response of bending–torsion coupled beams to deterministic and random loads. They solved the governing boundary value problem by classical modal analysis, where the geometric displacement co-ordinates (the lateral displacement and the angle of twist) are transformed to a single set of the modal amplitudes.

However, this procedure leads to solutions which are slowly convergent or even divergent. Hence, this paper introduces a different approach. Thereby, the dynamic response is separated into a quasistatic and a complementary dynamic response, and a modal expansion is performed only for the complementary dynamic part of the solution. The quasistatic portion is determined separately and in a closed form by means of weighted integration of the corresponding influence function. Such a splitting is numerically efficient and also more accurate since the quasistatic part may contain singularities or discontinuities that are properly accounted for and which would be poorly modelled by a truncated modal series solution. The remaining complementary dynamic response is non-singular and can be approximated by a finite modal series of fast accelerated convergence. This type of solution procedure has been first suggested by Boley and Barber [7] for the analysis of rapidly heated beams and plates and was later picked up by Ziegler *et al*. [8-11], in order to analyze the elastic-plastic behvior of beams and plates. The proposed procedure is illustrated for a simply supported beam with channel cross-sections, and the improvement in comparison to the classical modal analysis is shown.

## 2. GOVERNING EQUATIONS

In the present paper coupled bending and torsional vibrations of a thin-walled monosymmetric beam with open cross-section are considered, which consists of a linear elastic material with mass density $\rho$. The centroid and the shear-center are denoted by $S$ and $M$, respectively, and they are separated by the distance $c$ (see Figure 1). The beam is referenced to a Cartesian system of co-ordinates $x$, $y$, $z$, where the shear center axis is taken to be the $x$-axis and the $y$-axis coincides with the symmetry axis of the cross-section. Furthermore, it is assumed that the beam is loaded by a given transverse force per unit length $q(x, t)$ distributed along the centroidal axis and an external torque of intensity $m(x, t)$, Figure 1. The deformation is determined by the lateral deflection $w(x, t)$ of the shear center axis and by the angle of twist $\vartheta(x, t)$ of the cross-section. As, for example Weaver *et al*. [1] have shown, the dynamic response is governed by the following set of coupled differential equations of motion:

$$EJ_y w_{,xxxx} + \rho A(\ddot{w} + c\ddot{\vartheta}) = q, \tag{1a}$$

$$EA_{\varphi\varphi}\vartheta_{,xxxx} - GI_T\vartheta_{,xx} + \rho(I_0 + c^2A)\ddot{\vartheta} + \rho Ac\ddot{w} = m + cq, \tag{1b}$$

where $EJ_y$ is the bending rigidity of the beam with respect to the $y$-axis, $EA_{\varphi\varphi}$ denotes the warping rigidity, $GI_T$ refers to the torsional rigidity for uniform torsion, $I_0$ represents the centroidal polar moment of inertia of the cross-section, and $A$ is the cross-sectional area. $(\,)_{,x}$ and $(\dot{\,})$ stand for the spatial and time derivatives.

The solution of equations (1) depends on the initial conditions at time instant $t = 0$ and on the actual boundary conditions at point $x_b$. It is assumed that the beam is of length $l$ so that these conditions are imposed at the sections $x_b = 0$ and $x_b = l$. In the following, three classical boundary conditions are summarized, compare e.g., reference [3]:

(i) Simply supported end:

$$w(x_b, t) = 0, \qquad w_{,xx}(x_b\, t) = 0, \qquad \vartheta(x_b, t) = 0, \qquad \vartheta_{,xx}(x_b, t) = 0. \tag{2a}$$

(ii) Clamped end:

$$w(x_b, t) = 0, \qquad w_{,x}(x_b, t) = 0, \qquad \vartheta(x_b, t) = 0, \qquad \vartheta_{,x}(x_b, t) = 0. \tag{2b}$$

(iii) Free end:

$$w_{,xx}(x_b, t) = 0, \qquad w_{,xxx}(x_b, t) = 0, \qquad \vartheta_{,xx}(x_b, t) = 0,$$
$$-EA_{\varphi\varphi}\vartheta_{,xxx} + GI_T\vartheta_{,x} = 0. \tag{2c}$$

The free-vibration analysis of equations (1) can be found in reference [3].

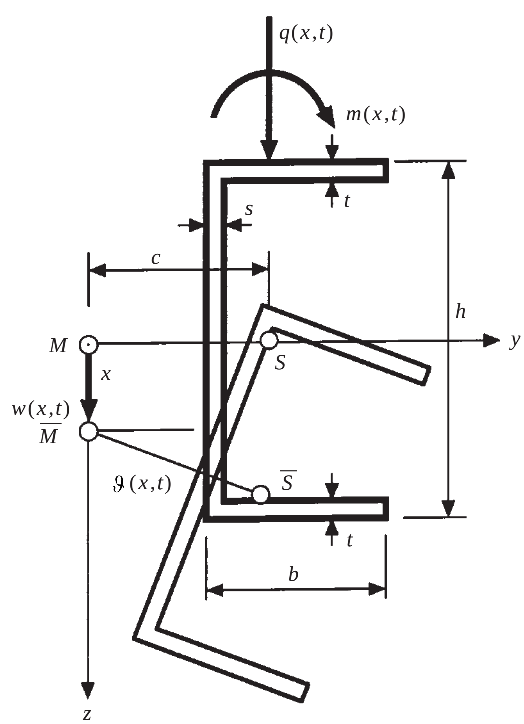


Figure 1. Channel cross-section

The bending moment $M_y$, the shear force $Q_z$, the bimoment $B_\varphi$, the primary torque $M_{T_p}$ and the torque of constrained twisting $M_{T\varphi}$ are related to the deflection and to the angle of twist as follows:

$$M_y = -EJ_y w_{,xx}, \qquad Q_z = -EJ_y w_{,xxx}, \tag{3a}$$

$$B_\varphi = -EA_{\varphi\varphi}\vartheta_{,xx}, \qquad M_{T_p} = GI_T\vartheta_{,x}, \qquad M_{T_\varphi} = -EA_{\varphi\varphi}\vartheta_{,xxx}. \tag{3b}$$

## 3. DYNAMIC RESPONSE ANALYSIS

Within a linear theory of beam structures the quasistatic response can always be represented in a closed form. The quasistatic part of the solution possibly contains singularities or discontinuities, whereas the remaining dynamic part of the solution is non-singular. Due to its smooth behavior, this remaining part can be described by means of a relatively small number of mode shapes. Consequently, the total response $w(x, t)$, $\vartheta(x, t)$ will be formulated as the sum of its analytic quasistatic part (denoted by a superscript $(\,)^S$) and a modal expansion of its complementary dynamic portion (denoted by a superscript $(\,)^D$) [7–11],

$$w(x, t) = w^S(x, t) + w^D(x, t), \qquad \vartheta(x, t) = \vartheta^S(x, t) + \vartheta^D(x, t). \tag{4}$$

Replacing $w(x, t)$ and $\vartheta(x, t)$ of equations (1) by the expressions of equations (4) renders:

$$EJ_y w^S_{,xxxx} + EJ_y w^D_{,xxxx} + \rho A\ddot{w}^S + \rho A\ddot{w}^D + \rho Ac\ddot{\vartheta}^S + \rho Ac\ddot{\vartheta}^D = q, \tag{5a}$$

$$EA_{\varphi\varphi}\vartheta^S_{,xxxx} + EA_{\varphi\varphi}\vartheta^D_{,xxxx} - GI_T\vartheta^S_{,xx} - GI_T\vartheta^D_{,xx} + \rho(I_0 + c^2A)\ddot{\vartheta}^S$$
$$+ \rho(I_0 + c^2A)\ddot{\vartheta}^D + \rho Ac\ddot{w}^S + \rho Ac\ddot{w}^D = m + cq. \tag{5b}$$

Considering the differential equations of the quasistatic response,

$$EJ_y w^S_{,xxxx} = q, \qquad EA_{\varphi\varphi}\vartheta^S_{,xxxx} - GI_T\vartheta^S_{,xx} = m + cq, \tag{6}$$

the equations of motion of the complementary dynamic response can be separated from equations (5):

$$EJ_y w^D_{,xxxx} + \rho A(\ddot{w}^D + c\ddot{\vartheta}^D) = -\rho A(\ddot{w}^S + c\ddot{\vartheta}^S), \tag{7a}$$

$$EA_{\varphi\varphi}\vartheta^D_{,xxxx} - GI_T\vartheta^D_{,xx} + \rho A\left[\left(\frac{I_0}{A} + c^2\right)\ddot{\vartheta}^D + c\ddot{w}^D\right]$$
$$= -\rho A\left[\left(\frac{I_0}{A} + c^2\right)\ddot{\vartheta}^S + c\ddot{w}^S\right]. \tag{7b}$$

The solution of equations (7) is found by modal analysis. Thereby, the quasistatic parts $w^S$, $\vartheta^S$ and the complementary dynamic portions $w^D$, $\vartheta^D$ are transformed to a set of modal amplitudes. These transformations are expressed as

$$w^S(x, t) = \sum_{n=1}^{\infty} Y^S_n(t)\Phi_n(x), \qquad \vartheta^S(x, t) = \sum_{n=1}^{\infty} Y^S_n(t)\Psi_n(x), \tag{8a}$$

$$w^D(x,t) = \sum_{n=1}^{\infty} Y_n^D(t)\Phi_n(x), \qquad \vartheta^D(x,t) = \sum_{n=1}^{\infty} Y_n^D(t)\Psi_n(x). \tag{8b}$$

The modal series equations (8) are inserted into equations (7), multiplied by $\Phi_m$ and $\Psi_m$, respectively, and added. Integration over the beam length $l$ and considering the orthonormality relation equation [12],

$$\int_0^l \left[\Phi_n\Phi_m + c(\Psi_n\Phi_m + \Phi_n\Psi_m) + \left(\frac{I_0}{A} + c^2\right)\Psi_n\Psi_m\right]\mathrm{d}x = \delta_{nm}, \tag{9}$$

leads to a formally decoupled system of single-degree-of-freedom oscillator equations for the complementary dynamic variables $Y_n^D(t)$:

$$\ddot{Y}_n^D + \omega_n^2 Y_n^D = -\ddot{Y}_n^S. \tag{10}$$

In equation (9) $\delta_{nm}$ denotes the Kronecker Delta function. In the next step, $Y_n^S$ is evaluated by the same procedure. Thereby, equations (8a) are inserted into equations (6), and after some algebra the quasistatic modal amplitudes become

$$Y_n^S(t) = \frac{1}{\rho A\omega_n^2} P_n(t), \tag{11}$$

where

$$P_n(t) = \int_0^l [q(x,t)\Phi_n(x) + m(x,t)\Psi_n(x) + cq(x,t)\Psi_n(x)]\,\mathrm{d}x \tag{12}$$

is the generalized loading associated with the mode shapes $\Phi_n$ and $\Psi_n$. Finally, the solution of equation (10) is given by means of Duhamel's convolution integral [13],

$$Y_n^D(t) = Y_n^D(0)\cos\omega_n t + \frac{\dot{Y}_n^D(0)}{\omega_n}\sin\omega_n t - \frac{1}{\rho A\omega_n^3}\int_0^t \ddot{P}_n(\tau)\sin[\omega_n(t-\tau)]\,\mathrm{d}\tau, \tag{13}$$

where $Y_n^D(0)$, $\dot{Y}_n^D(0)$ represent the initial conditions at $t = 0$:

$$Y_n^D(0) = Y_n(0) - \frac{1}{\rho A\omega_n^2} P_n(0), \tag{14a}$$

$$\dot{Y}_n^D(0) = \dot{Y}_n(0) - \frac{1}{\rho A\omega_n^2} \dot{P}_n(0). \tag{14b}$$

In equations (14) $Y_n(0)$, $\dot{Y}_n(0)$ are derived from the initial conditions at $t = 0$ by modal decomposition of $w(x,0)$, $\vartheta(x,0)$, $\dot{w}(x,0)$ and $\dot{\vartheta}(x,0)$ according to equations (8). Note, that in general $Y_n^D(0)$, $\dot{Y}_n^D(0)$ do not vanish also in case of quiet initial conditions. Details of how the convolution integral in equation (13) can be evaluated in the case of loading functions with arbitrary time history are given in Appendix A.

An additional convenient feature of that kind of modal approach is the incorporation of viscous damping, that is introduced via modal damping coefficients; see e.g., reference [14].

## 4. APPLICATION

The proposed procedure is applied to a simply supported beam. The free response analysis according to reference [3] renders the mode shapes

$$\Phi_n(x) = A_n \sin \lambda_n x, \qquad \Psi_n(x) = B_n \sin \lambda_n x, \qquad \lambda_n = \frac{n\pi}{l}, \tag{15}$$

and the corresponding eigenfrequencies

$$\omega_{n\pm}^2 = (I_0 + c^2 A)\left\{\bar{\Omega}_n^2 + \hat{\Omega}_n^2 \pm \left[(\bar{\Omega}_n^2 + \hat{\Omega}_n^2)^2 - 4\,\frac{I_0}{I_0 + c^2 A}\,\bar{\Omega}_n^2 \hat{\Omega}_n^2\right]^{0\cdot 5}\right\}/2/I_0, \tag{16}$$

with

$$\bar{\Omega}_n^2 = \lambda_n^4 \frac{EJ_y}{\rho A}, \qquad \hat{\Omega}_n^2 = \lambda_n^2 \frac{EA_{\varphi\varphi}\lambda_n^2 + GI_T}{\rho(I_0 + c^2 A)}. \tag{17}$$

Note, that for each $n$ two modes of vibration are obtained: a higher one $\omega_{n+}$ and a lower one $\omega_{n-}$. Coefficients $A_{n\pm}$, $B_{n\pm}$ are determined as follows:

$$A_{n\pm} = \sqrt{\frac{2}{l}}\left[1 + 2c\alpha_{n\pm} + \left(\frac{I_0}{A} + c^2\right)\alpha_{n\pm}\right]^{-1/2},$$

$$B_{n\pm} = \alpha_{n\pm} A_{n\pm}, \qquad \alpha_{n\pm} = \frac{\bar{\Omega}_n^2 - \omega_{n\pm}^2}{c\omega_{n\pm}^2}. \tag{18}$$

In the following, results obtained by the proposed procedure are compared with those derived by means of the classical modal analysis [6]. The latter represents the total deflection and the total angle of twist,

$$w(x, t) = \sum_{n=1}^{\infty} Y_n(t)\Phi_n(x), \qquad \vartheta(x, t) = \sum_{n=1}^{\infty} Y_n(t)\Psi_n(x), \tag{19}$$

where the modal coefficients are of the form

$$Y_n(t) = Y_n(0)\cos \omega_n(t) + \frac{\dot{Y}_n(0)}{\omega_n}\sin \omega_n t + \frac{1}{\rho A \omega_n}\int_0^t P_n(\tau)\sin[\omega_n(t-\tau)]\,\mathrm{d}\tau. \tag{20}$$

At time $t = 0$ a single time-harmonic lateral force at midspan, $q(x, t) = F_0\delta(x - l/2) \times \sin \nu t$, is switched on. $\delta(x - l/2)$ denotes the Dirac delta function at $x = l/2$. In that particular case, the quasistatic deflection and the quasistatic angle of twist read [15]:

$$w^S(x, t) = \frac{F_0 l^3}{48EJ_y}\,\xi(3 + 4\xi^2)\sin \nu t, \qquad 0 \leqslant x \leqslant l/2, \tag{21a}$$

$$w^S(x, t) = \frac{F_0 l^3}{48EJ_y}\,\bar{\xi}(3 + 4\bar{\xi}^2)\sin \nu t, \qquad l/2 \leqslant x \leqslant l, \tag{21b}$$

$$\vartheta^S(x, t) = \frac{F_0 c l}{2GI_T}\left[\xi - \frac{2\sinh(0{\cdot}5\varepsilon)\sinh(\varepsilon\xi)}{\varepsilon\sinh\varepsilon}\right]\sin \nu t, \qquad 0 \leqslant x \leqslant l/2, \tag{21c}$$

$$\vartheta^S(x, t) = \frac{F_0 c l}{2GI_T}\left[\bar{\xi} - \frac{2\sinh(0{\cdot}5\varepsilon)\sinh(\varepsilon\bar{\xi})}{\varepsilon\sinh\varepsilon}\right]\sin \nu t, \qquad l/2 \leqslant x \leqslant l, \tag{21d}$$

with

$$\varepsilon = l\sqrt{\frac{GI_T}{EA_{\varphi\varphi}}}, \qquad \xi = \frac{x}{l}, \qquad \bar{\xi} = 1 - \frac{x}{l}. \tag{22}$$

The corresponding quasistatic internal actions are given by equations (3), when substituting $w^S$ for $w$, $\vartheta^S$ for $\vartheta$ and their derivatives.

The dynamic analysis according to equations (5), (8) and (13) renders the complementary dynamic quantities:

$$w^D(x, t) = \frac{F_0\nu}{\rho A}\sum_{n=1,3,5\ldots}^{\infty}\sin\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n-}}{\omega_{n-}^3}g_{n-}(t) + \frac{\mathscr{L}_{n+}}{\omega_{n+}^3}g_{n+}(t)\right], \tag{23a}$$

$$\vartheta^D(x, t) = \frac{F_0\nu}{\rho A}\sum_{n=1,3,5\ldots}^{\infty}\sin\frac{n\pi x}{l}\left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}^3}g_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}^3}g_{n+}(t)\right], \tag{23b}$$

$$M_y^D(x, t) = \frac{F_0\nu EJ_y}{\rho A}\sum_{n=1,3,5\ldots}^{\infty}\left(\frac{n\pi}{l}\right)^2\sin\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n-}}{\omega_{n-}^3}g_{n-}(t) + \frac{\mathscr{L}_{n+}}{\omega_{n+}^3}g_{n+}(t)\right], \tag{23c}$$

$$Q_z^D(x, t) = \frac{F_0\nu EJ_y}{\rho A}\sum_{n=1,3,5\ldots}^{\infty}\left(\frac{n\pi}{l}\right)^3\cos\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n-}}{\omega_{n-}^3}g_{n-}(t) + \frac{\mathscr{L}_{n+}}{\omega_{n+}^3}g_{n+}(t)\right], \tag{23d}$$

$$B_\varphi^D(x, t) = \frac{F_0\nu EA_{\varphi\varphi}}{\rho A}\sum_{n=1,3,5\ldots}^{\infty}\left(\frac{n\pi}{l}\right)^2\sin\frac{n\pi x}{l}\left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}^3}g_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}^3}g_{n+}(t)\right], \tag{23e}$$

$$M_{Tp}^{D}(x, t) = \frac{F_0 v G I_T}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \frac{n\pi}{l} \cos\frac{n\pi x}{l}\left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}^3} g_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}^3} g_{n+}(t)\right], \tag{23f}$$

$$M_{T\varphi}^{D}(x, t) = \frac{F_0 v E A_{\varphi\varphi}}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{n\pi}{l}\right)^3 \times \cos\frac{n\pi x}{l}\left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}^3} g_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}^3} g_{n+}(t)\right]. \tag{23g}$$

$\mathscr{L}_n$ is the $n$th participation factor due to a single force at midspan,

$$\mathscr{L}_{n\pm} = -A_{n\pm}^2(1 + c\alpha_{n\pm})(-1)^{(n+1)/2}, \tag{24}$$

and $g_n(t)$ denotes the $n$th complementary dynamic unit response due to a sinusoidal force excitation,

$$g_{n\pm}(t) = \frac{v}{v^2 - \omega_{n\pm}^2}(v \sin\omega_{n\pm} t - \omega_{n\pm} \sin vt) - \sin\omega_{n\pm} t. \tag{25}$$

In contrast, the results according to the classical modal analysis, equations (19), (20), become:

$$w(x, t) = \frac{F_0}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \sin\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{26a}$$

$$\vartheta(x, t) = \frac{F_0}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \sin\frac{n\pi x}{l}\left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{26b}$$

$$M_y(x, t) = \frac{F_0 E J_y}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t) + \frac{\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t)\right], \tag{26c}$$

$$Q_z(x, t) = \frac{F_0 E J_y}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{n\pi}{l}\right)^3 \cos\frac{n\pi x}{l}\left[\frac{\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{26d}$$

$$B_\varphi(x, t) = \frac{F_0 E A_{\varphi\varphi}}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} \left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{23c}$$

$$M_{Tp}(x, t) = \frac{F_0 G I_T}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \frac{n\pi}{l} \cos\frac{n\pi x}{l} \left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{26f}$$

$$M_{T\varphi}(x, t) = \frac{F_0 E A_{\varphi\varphi}}{\rho A} \sum_{n=1,3,5\ldots}^{\infty} \left(\frac{n\pi}{l}\right)^3 \cos\frac{n\pi x}{l} \left[\frac{\alpha_{n-}\mathscr{L}_{n-}}{\omega_{n-}} f_{n-}(t) + \frac{\alpha_{n+}\mathscr{L}_{n+}}{\omega_{n+}} f_{n+}(t)\right], \tag{26g}$$

with

$$f_{n\pm}(t) = \frac{1}{\nu^2 - \omega_{n\pm}^2} (\nu \sin \omega_{n\pm} t - \omega_{n\pm} \sin \nu t). \tag{27}$$

In order to show the improvement of the proposed procedure compared to the classical modal analysis the rate of convergence of the series solutions equations (26) and of the corresponding complementary dynamic series equations (23) is checked. Thereby, the series are approximated by a finite number of $N$ modes ($1 \leqslant N \leqslant 51$) and the error according to, e.g., for the deflection,

$$error(N) = \left|\frac{(w^S + w^{D,N}) - w_{ref}}{w_{ref}}\right| \times 100, \qquad error(N) = \left|\frac{w^N - w_{ref}}{w_{ref}}\right| \times 100, \tag{28}$$

is calculated for each number of $N$. In equations (28) the superscript $(\,)^N$ denotes the number of modes used in the approximation of the corresponding expression, and $w_{ref}$ is the reference solution, where the number of modes is chosen to be $N = 999$. This solution is referred as "exact", since the results determined numerically by both solution procedures are in agreement. Equations (21), (23) and (26) are evaluated for a beam with channel cross-section according to Figure 1 at time instant $t/T_{1-} = 1{\cdot}69$ ($T_{1-} = 2\pi/\omega_{1-}$) and an excitation frequency of $\nu = 4\omega_{1-}$. The geometric and mechanical properties of the beam are characterized by the following parameters: $l = 2{\cdot}2$ m, $h = 200$ mm, $b = 80$ mm, $t = 6$ mm, $s = 6$ mm, $E = 210\,000$ MN/m$^2$, $G = 80\,000$ MN/m$^2$, $\rho = 7850$ kg/m$^3$.

Figures 2–8 show the error of the deformation and of the corresponding internal actions at the specified point $x$ (left support or midspan) computed by means of the proposed procedure and classical modal analysis. It can be observed that the separate treatment of quasistatic and complementary dynamic response accelerates the rate of convergence. Especially the series solutions for the internal actions according to equations (26) are slowly convergent compared to solutions of equations (21) and (23).

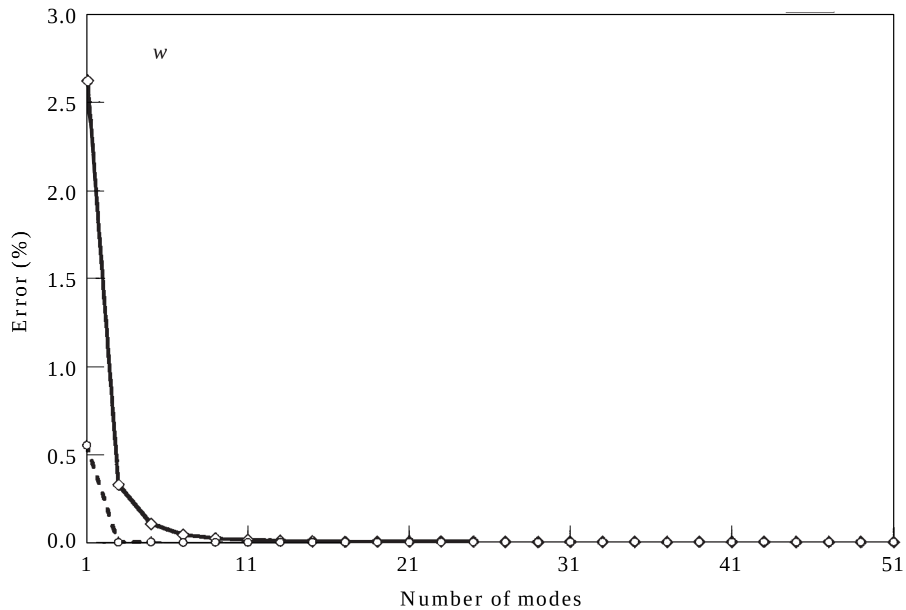


Figure 2. Convergence of the deflection at midspan. $v = 4\omega_{1-}$, $b/T_{1-} = 1{\cdot}69$, $x = l/2$. –◇–, Classical modal analysis; - - ○ - - , proposed analysis.

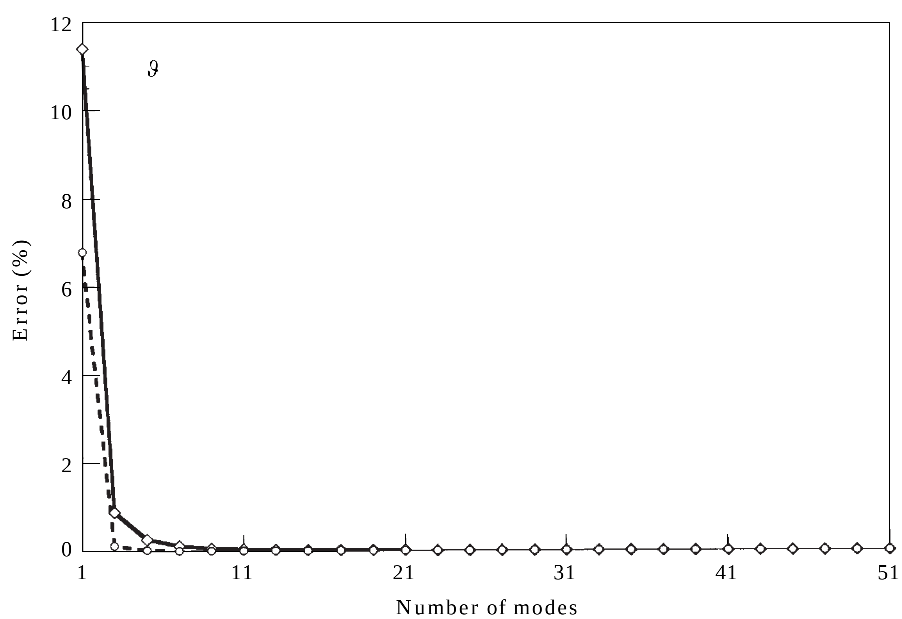


Figure 3. Convergence of the angle of twist at midspan. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = l/2$. Key as for Figure 2.

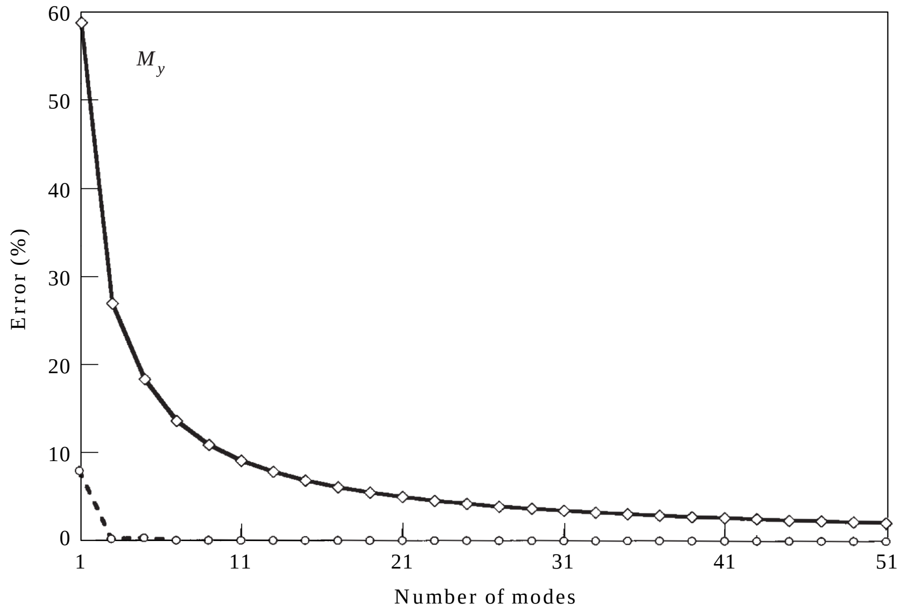


Figure 4. Convergence of the bending moment at midspan. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = l/2$. Key as for Figure 2.

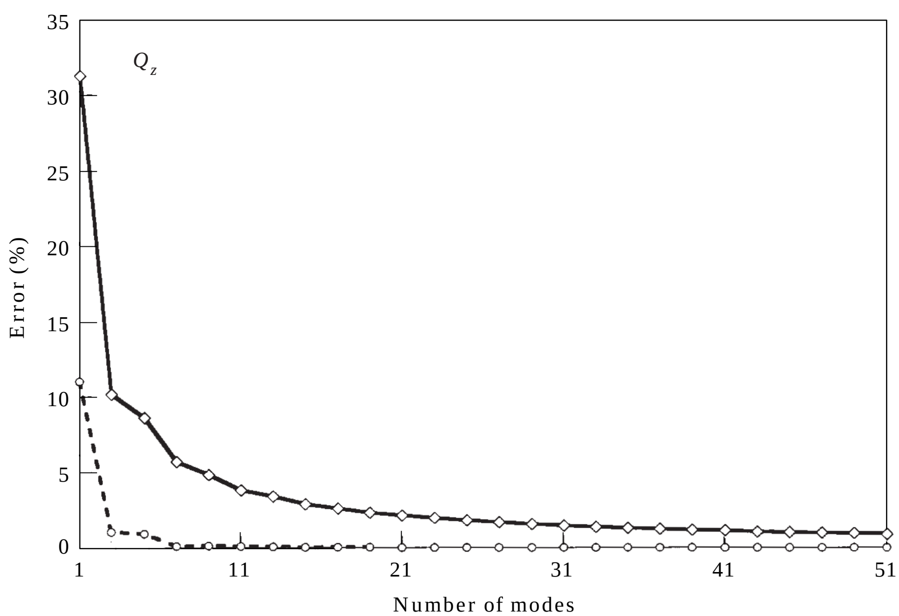


Figure 5. Convergence of the shear force at the left support. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = 0$. Key as for Figure 2.

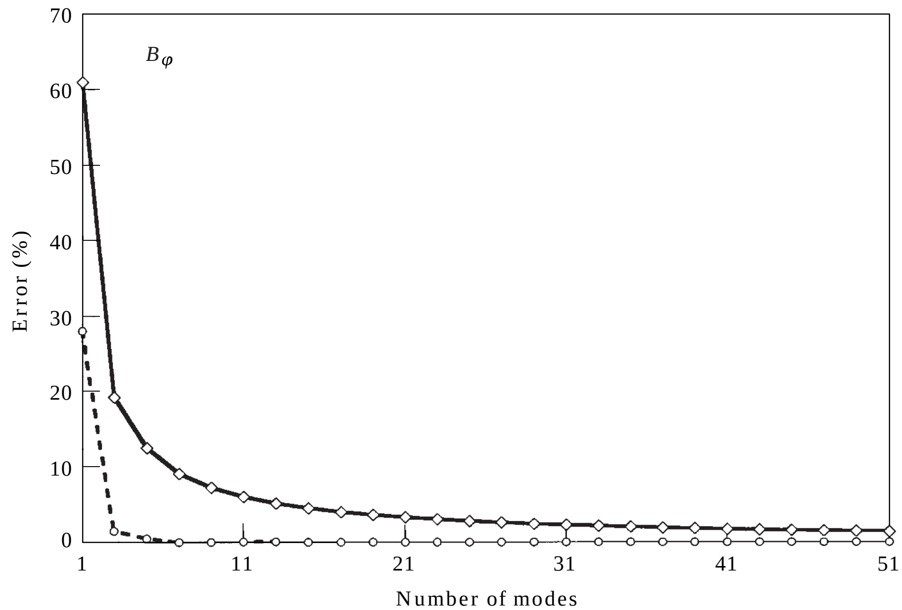


Figure 6. Convergence of the bimoment at midspan. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = l/2$. Key as for Figure 2.

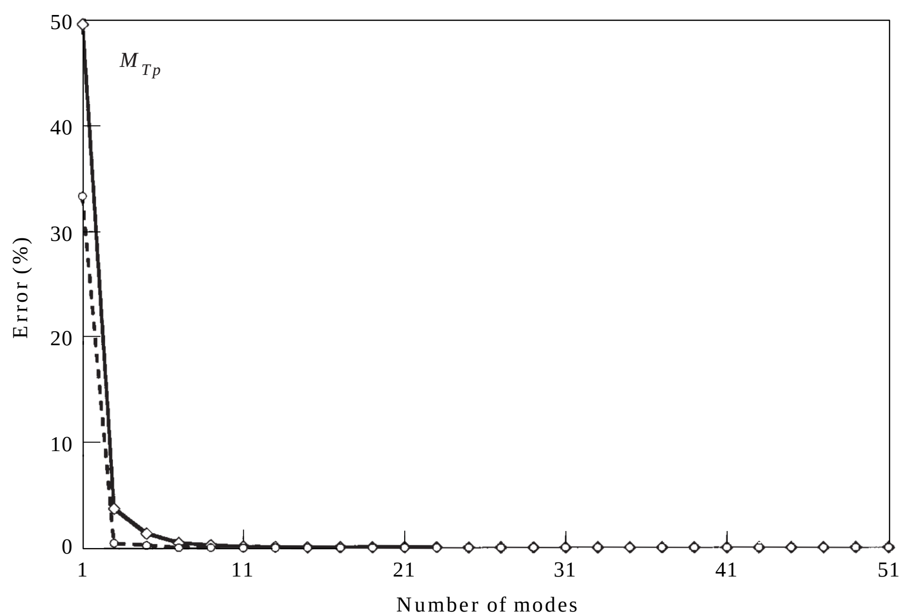


Figure 7. Convergence of the primary torque at the left support. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = 0$. Key as for Figure 2.

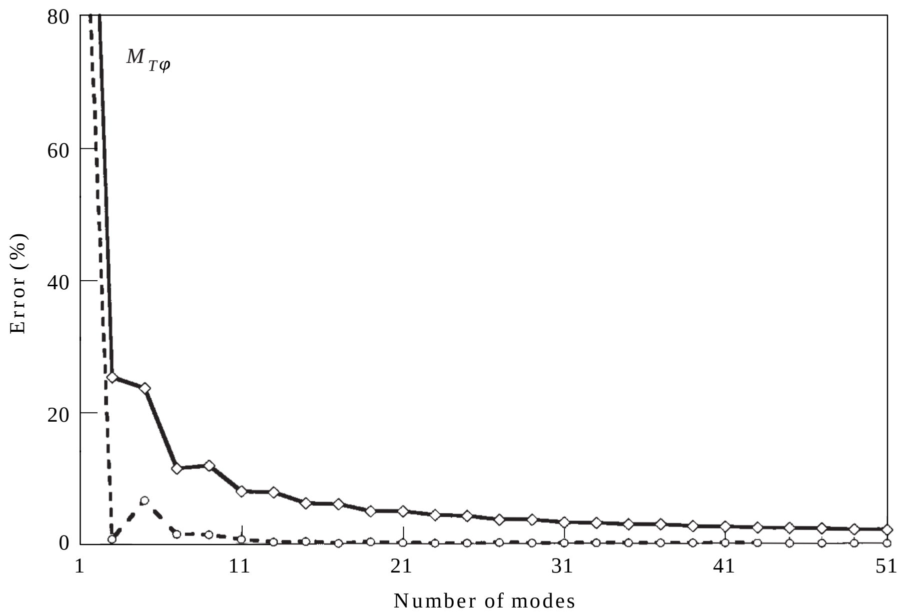


Figure 8. Convergence of the torque of constraint twisting at the left support. $v = 4\omega_{1-}$, $t/T_{1-} = 1{\cdot}69$, $x = 0$. Key as for Figure 2.

## 5. CONCLUSIONS

The initial-boundary value problem of coupled bending-torsional vibrations of elastic monosymmetric beams is solved. The solution of the corresponding set of partial equations of motion is found by separating the response of the beam in a quasistatic and in a complementary dynamic part. The quasistatic portion that probably contains singularities can be represented in a closed form. The remaining non-singular complementary dynamic part is approximated by a truncated modal series of fast accelerated convergence. The solution of the resulting generalized decoupled single-degree-of-freedom oscillators is given by Duhamel's convolution integral. An example is given for a simply supported composite beam under harmonic excitation. The procedure described in detail shows high improvement when compared to the classical modal analysis approach.

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## APPENDIX A: NUMERICAL EVALUATION OF THE DYNAMIC RESPONSE

In the case of loading functions with arbitrary time history, the solution is found incrementally. The convolution integral equation (13) is evaluated by assuming a linear variation of the load variables $q$ and $m$ within the time increment $\Delta t = t_{a+1} - t_a$,

$$q(x,t) = q(x,t_a) + \Delta q(x) g(\bar{t}), \qquad m(x,t) = m(x,t_a) + \Delta m(x) g(\bar{t}), \qquad \bar{t} = t - t_a, \tag{A1}$$

$$g(\bar{t}) = 1, \quad \bar{t} \geqslant \Delta t; \qquad g(\bar{t}) = \bar{t}/\Delta t, \quad 0 \leqslant \bar{t} \leqslant \Delta t; \qquad g(\bar{t}) = 0, \quad \bar{t} \leqslant 0. \tag{A2}$$

In the following, subscripts $(\,.\,)_a$ and $(\,.\,)_{a+1}$ refer to variables at the beginning and at the end of the time step, respectively. When computing the dynamic modal response according to equation (13), the time derivative $\ddot{g}$ enters, and, hence the approximate variation of $g(\bar{t})$ within the time interval must have a unique second derivative at $t_a$ and $t_{a+1}$. This degree of smoothness is achieved by assuming at least the linear ramp function $g(\bar{t})$ to start at $t_a^-$ immediately after $t_a$, and to end at $t_{a+1}^-$ immediately before $t_{a+1}$. Accordingly, the second derivative of $\Delta q$ and $\Delta m$ become

$$\Delta\ddot{q} = \Delta q(x)[\delta(0) - \delta(\Delta t)]/\Delta t, \qquad \Delta\ddot{m} = \Delta m(x)[\delta(0) - \delta(\Delta t)]/\Delta t, \tag{A3}$$

with Dirac delta function $\delta$. The evaluation of equation (13) together with equation (A3) renders the increments of the complementary dynamic coefficients [14],

$$\Delta Y_n^D = \dot{\mathscr{T}}_n Y_n^D(t_a) + \mathscr{T}_n \dot{Y}_n^D(t_a) - \frac{1}{\omega_n^2} \mathscr{D}_n \Delta P_n, \tag{A4}$$

with the following abbreviations:

$$\dot{\mathscr{T}}_n = \cos\omega_n \Delta t - 1, \qquad \mathscr{T}_n = \frac{1}{\omega_n} \sin\omega_n \Delta t, \qquad \mathscr{D}_n = \frac{1}{\omega_n} \frac{1}{\Delta t} \sin\omega_n \Delta t. \tag{A5}$$

$\Delta P_n$ stands for the increment of the generalized load,

$$\Delta P_n = \int_0^l [\Delta q(x)\Phi_n(x) + \Delta m(x)\Psi_n(x) + c\Delta q(x)\Psi_n(x)]\,\mathrm{d}x. \tag{A6}$$

Relation equation (A4) has to be completed by the increments of the velocity of modal coefficients. They are given by [14],

$$\Delta \dot{Y}_n^D = \dot{\mathscr{J}}_n Y_n^D(t_a) + \mathscr{J}_n \dot{Y}_n^D(t_a) - \frac{1}{\omega_n^2} \dot{\mathscr{D}}_n \Delta P_n, \tag{A7}$$

with

$$\dot{\mathscr{J}}_n = -\omega_n \sin\omega_n \Delta t, \qquad \mathscr{J}_n = \cos\omega_n \Delta t - 1, \qquad \dot{\mathscr{D}}_n = \frac{1}{\Delta t}(\cos\omega_n \Delta t - 1). \tag{A8}$$

Adding the exactly computed quasistatic increments to the complementary dynamic increments leads to the total dynamic response,

$$\Delta w(x) = \Delta w^S(x) + \sum_{n=1}^{N} \Phi_n(x)\Delta Y_n^D, \qquad \Delta\psi(x) = \Delta\psi^S(x) + \sum_{n=1}^{N} \Psi_n(x)\Delta Y_n^D, \tag{A9}$$

whereby the infinite series are approximated by a finite number of $N$ mode shapes.